An Introduction to Spectral Graph Theory

Sam Spiro, UC San Diego.

The Adjacency Matrix

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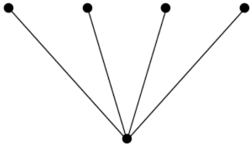

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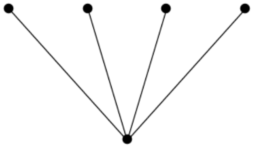

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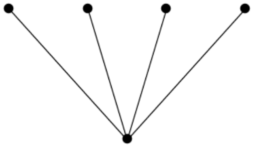

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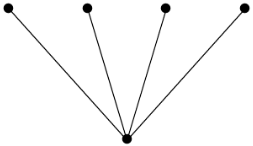

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A priori, A is just a table of numbers representing G, and in particular there's no reason to expect that its structure as a linear operator encodes anything about G. Remarkably this is not the case! Because A is a real symmetric matrix, it has real eigenvalues $\mu_1 \geq \cdots \geq \mu_n$.

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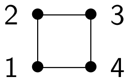

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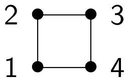

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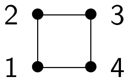

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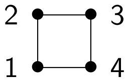

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$$(Ay)_u = \sum_{v \sim u} y_v$$

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Thus $Ay = -\mu y$.

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$$(Ay)_u = \sum_{v \sim u} y_v = -\sum_{v \sim u} x_v = -(Ax)_u = -\mu x_u = -\mu y_u.$$

Thus $Ay = -\mu y$. Further, k linearly independent eigenvectors of μ correspond to k linearly independent eigenvectors of $-\mu$, so the spectrum of A is symmetric about 0.

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A graph G is bipartite iff $\sigma(A)$ is symmetric about 0.

Conversely, if the spectrum of A is symmetric about 0, then $\sum \mu_i^k = 0$ for all odd k.

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Theorem (Hoffman; Wilf)

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Theorem (Hoffman)

If G is d-regular then $\alpha(G) \leq \frac{-\mu_n}{d-\mu_n} \cdot n$.

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If G has maximum degree Δ and average degree $\bar{d}$. Then

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This gives $|\mu_1| \leq \Delta$, which in particular implies the result. □

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The Sensitivity Conjecture

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More precisely it's the graph whose adjacency matrix can be defined recursively as $A_0 = [0]$ and

$$A_n = \begin{bmatrix} A_{n-1} & I \\ I & A_{n-1} \end{bmatrix}.$$

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We recall that an induced subgraph of G is any graph that is obtained by selecting some subset of its vertices and then taking every edge of G involving vertices of this subset.

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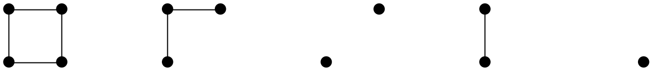

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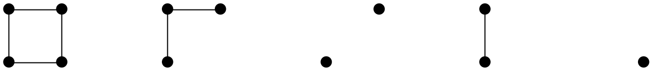

Theorem (Chung, Füredi, Graham, Seymour 1988)

If H is an induced subgraph of Q_n on $2^{n-1}+1$ vertices, then H has maximum degree at least $\frac{1}{2}\log_2(n)$.

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Theorem (Huang 2019)

If H is an induced subgraph of Q_n on $2^{n-1}+1$ vertices, then it has maximum degree at least $\sqrt{n}$.

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The Sensitivity Conjecture

Lemma (Cauchy Interlacing Theorem)

Let B be a real symmetric $n \times n$ matrix and C an $m \times m$ principal sumbmatrix of B with $m \leq n$. If B has eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and C has eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, then

$$\lambda_i \geq \mu_i \geq \lambda_{i+n-m}.$$

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Corollary

From the eigenvalues of A it is impossible to determine if G is connected, contains a C_4, etc.

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For this to actually be useful, it is crucial that the matrix (and its eigenvalues) are reasonable to compute. E.g. the following is not a very useful matrix

$$X_G = \chi(G) \cdot I.$$

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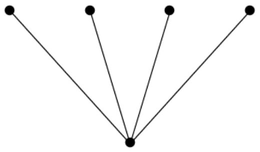

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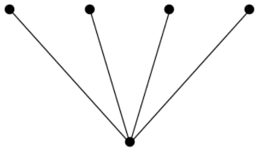

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Note that L arises as a boundary-coboundary operator, as well as a chip firing operator in the Abelian sandpile model.

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Corollary (Cayley's Formula)

The number of labeled trees on n vertices is n^{n-2}.

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Theorem (Godsil, Newman)

Let S be an independent set in G. If $\bar{d}(S)$ is the average degree of the vertices in S, then

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because $D = dI$ (giving $L = dI - A$).
More generally, for regular graphs it is often the case that many choices of M will solve the problem (with the "correct M" generalizing to non-regular graphs).

The Normalized Laplacian

Let D be the diagonal matrix of degrees of G. For G a graph without isolated vertices, define the normalized Laplacian matrix

$$\mathcal{L} = D^{-1/2} L D^{-1/2} = I - D^{-1/2} A D^{-1/2}.$$

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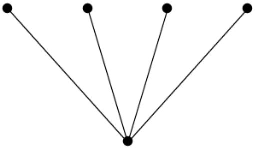

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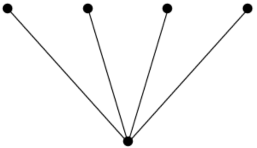

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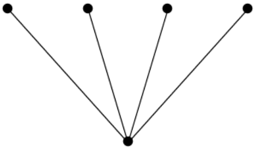

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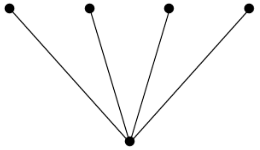

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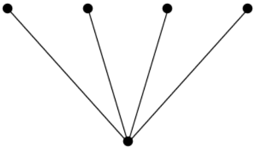

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How’d This Go?

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Link also on my website.

The End

Thank You!